J. Differential Equations 250 (2011) 223–242

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Sobolev type embedding and quasilinear elliptic equations with radial potentials

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ARTICLE INFO

Article history:
Received 26 November 2009
Revised 20 August 2010

MSC:
35J05
35J20
35J60
58C20

Keywords:
Quasilinear elliptic equations
Unbounded or decaying potentials
Sobolev type embedding

ABSTRACT

We study the existence of nontrivial radial solutions for quasilinear elliptic equations with unbounded or decaying radial potentials. The existence results are based upon several new embedding theorems we establish in the paper for radially symmetric functions.

1. Introduction

In this paper we study the existence of entire solutions of nonlinear elliptic equations of the form

$$\begin{cases} -\operatorname{div}\left(A\left(|x|\right)|\nabla u|^{p-2}\nabla u\right) + V\left(|x|\right)|u|^{p-2}u = Q\left(|x|\right)f(u) & \text{in } \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $p > 1$ and the potentials involved are radially symmetric and may be singular (unbounded, decaying and vanishing). In particular, we are interested in the effect on the solutions structure from the interplay between the quasilinear nature $p > 1$, the nonlinear growth of $f(u)$, and the singularities

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1 Supported by NSFC-10831005, NSFB-1082004, KZ201010028027 and SRFDP20070028004.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2010.08.025

of the potentials A, V, and Q. The motivation for our study here is twofold. First of all, there are the classical works in [7,18,23,24,28,33,34] which give stronger results for existence of radially symmetric solutions, and classical works in [20–22,34] which assert stronger embedding results for functions in radially symmetric class. There has been a revived interest in recent years to further explore along this line of work (e.g., [5,8,13,14,29,31,32,35]). In this paper, in particular, we would like to examine the effects of the singularities and vanishing of the potentials A, V, and Q on the ranges of embedding and on the existence of solutions. Secondly, in the recent study of semiclassical states of nonlinear Schrödinger type equations people have tried to investigate the effects of the singular and decaying potentials. Some interesting work has been done such as in [1,2,4,6,8–11,17,19,25–27,30]. In this paper on one hand we extend some of the recent embedding results for radially symmetric functions to more general cases allowing in particular weighted p-Laplacian operator involved here; on the other hand our results further reveal relations between the growth and decay of the potentials and the ranges of embedding. This may be useful information in the study of semiclassical states of nonlinear Schrödinger equations. As applications of our embedding results we establish a general existence result for Eq. (1.1).

We assume that the weight functions A, V and Q satisfy the following assumptions.

(A) $A \in C(0,\infty)$, $A(r) > 0$ and there exist real numbers $\ell, \ell_0 > p - N$ and $c_0 > 0$, $c_\infty > 0$ such that

$$\lim_{r\to\infty} \frac{A(r)}{r^{\ell}} = c_\infty, \qquad \lim_{r\to 0} \frac{A(r)}{r^{\ell_0}} = c_0.$$

(V) $V \in C(0,\infty)$, $V(r) > 0$ and there exist real numbers a and a_0 such that

$$\liminf_{r\to\infty} \frac{V(r)}{r^{a}} > 0, \qquad \liminf_{r\to 0} \frac{V(r)}{r^{a_0}} > 0.$$

(Q) $Q \in C(0,\infty)$, $Q(r) > 0$ and there exist real numbers b and b_0 such that

$$\limsup_{r\to\infty} \frac{Q(r)}{r^{b}} < \infty, \qquad \limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty.$$

The existence and embedding results depend on the behaviors of these potentials near 0 and near infinity. Let us define the index q_* according to the relations between p, ℓ and a, b by

$$q_* = \begin{cases} \frac{p^2(N-1+b)+p(\ell-a)}{p(N-1+a)+(\ell-a)}, & b \geqslant a > -p+\ell, \\ \frac{p(N+b)}{N+\ell-p}, & b \geqslant -p+\ell,\ a \leqslant -p+\ell, \\ p, & b \leqslant \max\{a, -p+\ell\}, \end{cases}$$

and define the indices q^*, q_{**} according to the relations between p, ℓ_0 and a_0, b_0 by

$$q^* = \begin{cases} \frac{p(N+b_0)}{N+\ell_0-p}, & b_0 \geqslant -p+\ell_0,\ a_0 \geqslant -p+\ell_0, \\ \frac{p^2(N-1+b_0)+p(\ell_0-a_0)}{p(N-1+a_0)+(\ell_0-a_0)}, & -p+\ell_0 \geqslant a_0 > -\frac{p(N-1)+\ell_0}{p-1},\ b_0 \geqslant a_0, \\ \infty, & a_0 \leqslant -\frac{p(N-1)+\ell_0}{p-1},\ b_0 \geqslant a_0, \end{cases}$$

$$q_{**} = \frac{p^2(N-1+b_0)+p(\ell_0-a_0)}{p(N-1+a_0)+(\ell_0-a_0)}, \quad b_0 \leqslant a_0 < -\frac{p(N-1)+\ell_0}{p-1}.$$

We introduce some notations of function spaces. Let $C_0^\infty(\mathbb{R}^N)$ denote the collection of smooth functions with compact support and

$$C_{0,r}^\infty(\mathbb{R}^N) = \{u \in C_0^\infty(\mathbb{R}^N) \mid u \text{ is radial}\}.$$

Let $D_r^{1,p}(\mathbb{R}^N; A)$ be the completion of $C_{0,r}^\infty(\mathbb{R}^N)$ under

$$\|u\|_A = \left(\int_{\mathbb{R}^N} A(|x|)|\nabla u|^p\,dx\right)^{\frac{1}{p}}.$$

Indeed, by similar arguments to that used in [15] we have that

$$D_r^{1,p}(\mathbb{R}^N; A) = \overline{C_{0,r}^\infty(\mathbb{R}^N)}^{\|\cdot\|_A} = \overline{C_{0,r}^\infty(\mathbb{R}^N \setminus \{0\})}^{\|\cdot\|_A}.$$

Define for $p \geqslant 1$

$$L^p(\mathbb{R}^N; V) = \left\{u\colon \mathbb{R}^N \to \mathbb{R} \;\middle|\; u \text{ is Lebesgue measurable}, \int_{\mathbb{R}^N} V(|x|)|u|^p\,dx < \infty\right\}.$$

Then define

$$W_r^{1,p}(\mathbb{R}^N; A, V) = D_r^{1,p}(\mathbb{R}^N; A) \cap L^p(\mathbb{R}^N; V)$$

which is, under conditions (A) and (V), a Banach space with the norm

$$\|u\|_{A,V} = \left(\int_{\mathbb{R}^N} A(|x|)|\nabla u|^p + V(|x|)|u|^p\,dx\right)^{\frac{1}{p}}.$$

When (Q) is assumed, we also define for $q \geqslant 1$

$$L^q(\mathbb{R}^N; Q) = \left\{u\colon \mathbb{R}^N \to \mathbb{R} \;\middle|\; u \text{ is Lebesgue measurable}, \int_{\mathbb{R}^N} Q(|x|)|u|^q\,dx < \infty\right\}.$$

In Sections 2 and 3 we will first establish various embedding results from $W_r^{1,p}(\mathbb{R}^N; A, V)$ into $L^q(\mathbb{R}^N; Q)$ with the range of q depending upon the various indices we defined above. Using these embedding results we can establish some existence results on Eq. (1.1). To state such a result we make the following assumptions on f.

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$, $f(0) = 0$.
(f_2) There is $\mu > q_* \geqslant p$ such that

$$0 < \mu F(u) := \mu \int_0^u f(t)\,dt \leqslant u f(u), \quad \text{for } u \in \mathbb{R} \setminus \{0\}.$$

(f_3) There is $C > 0$, $q_* < q_1 \leqslant q_2 < q^*$ (or $\max\{q_*, q_{**}\} < q_1 \leqslant q_2 < \infty$) such that

$$|f(u)| \leqslant C\left(|u|^{q_1-1} + |u|^{q_2-1}\right).$$

(f_4) $f(-u) = -f(u)$.
(f_5) There are $C > 0$ and $\tau > p$ such that $F(u) \geqslant C|u|^\tau$ for $u \in \mathbb{R}$.

The existence result in this paper is the following theorem.

Theorem 1.1. *Assume* (A), (V), (Q) *and* (f_1)–(f_3). *Then* (1.1) *has a nontrivial solution in* $W_r^{1,p}(\mathbb{R}^N; A, V)$. *Furthermore, if* (f_4) *and* (f_5) *are satisfied, then* (1.1) *has infinitely many solutions in* $W_r^{1,p}(\mathbb{R}^N; A, V)$.

We close up this section with a few further remarks and comments. This paper is a continuation of our earlier works [31] and [32]. In [31], we developed a unique variational framework for (1.1) with $p = 2$, $A = 1$ and $f(u) = |u|^{q-2}u$ by establishing some embedding theorems of Sobolev type for radial functions and unified many known partial results for the existence of ground state solutions which was established by a variety of different methods (e.g., [5,13,14,18,23,24,29,33–35]). In [32], we extended and developed the ideas and techniques in [31] to (1.1) with $p \neq 2$, $A = 1$ and $f(u) = |u|^{q-2}u$ and established more general Sobolev type embedding results in radial function spaces with potential which may be unbounded, decaying and vanishing. In the current paper, we deal with the existence of nontrivial solutions for the singular quasilinear elliptic equations by building more general type of weighted Sobolev embedding. The embedding results here improve the ones in [31,32] to including applications to anisotropic type equations. The paper is organized as follows. In Section 2 we give some inequalities involving radial functions, extending some known inequalities to more general cases; in Section 3 we establish some Sobolev type embedding results which extend the result in [31,32], and these results may be of independent interests. In Section 4 we use the embedding results to study solutions of Eq. (1.1).

2. Some inequalities involving radial functions

In this section we prove some inequalities on radial functions which are of their own interests. For any set $A \subset \mathbb{R}^N$, A^c denotes the complement of A. Denote by B_r the ball in $\mathbb{R}^N$ centered at 0 with radius r.

Lemma 2.1 *(Weighted version of the radial lemma). Let* $\alpha \in \mathbb{R}$ *be such that* $1 < p < N + \alpha$. *Then for all* $u \in D_r^{1,p}(\mathbb{R}^N, |x|^\alpha)$, *the completion of* $C_{0,r}^\infty(\mathbb{R}^N)$ *under*

$$\|u\|_{r,\alpha} := \left(\int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^p \, dx \right)^{\frac{1}{p}},$$

it holds

$$\bigl|u(x)\bigr| \leqslant \widehat{C} |x|^{-\frac{N+\alpha-p}{p}} \|u\|_{r,\alpha}, \tag{2.1}$$

where

$$\widehat{C} := \widehat{C}(N, p, \alpha) = \omega_N^{-\frac{1}{p}} \left(\frac{p-1}{N+\alpha-p} \right)^{\frac{p-1}{p}},$$

and ω_N *denotes the* $(N-1)$*-dimensional measure of the unit sphere in* $\mathbb{R}^N$.

The proof is similar to that in [32] we omit here. Lemma 2.1 extends the result in [32] which was for $\alpha = 0$. The case $\alpha = 0$, $p = 2$ (known as Ni's inequality) was due to [23] where this inequality was given in the unit ball of $\mathbb{R}^N$.

Lemma 2.2 *(Weighted Hardy inequality). Let $1 < p < N + \alpha$. Assume $p \leqslant q < \infty$ and write $q = \frac{p(N+c)}{N+\alpha-p}$ for some $\alpha - p \leqslant c < \infty$. Then there exists $\widetilde{C} > 0$ such that for all $u \in D_r^{1,p}(\mathbb{R}^N, |x|^\alpha)$*

$$\left(\int_{\mathbb{R}^N} |x|^c |u|^q \, dx\right)^{\frac{p}{q}} \leqslant \widetilde{C} \int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^p \, dx. \tag{2.2}$$

Proof. By a standard density argument, it suffices to consider the case that $u \in C_{0,r}^\infty(\mathbb{R}^N)$. For such a u, by using Lemma 2.1, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |x|^c |u|^q \, dx &= \omega_N \int_0^\infty r^{N-1+c} \big|u(r)\big|^q \, dr \\
&= -\frac{q\omega_N}{N+c} \int_0^\infty r^{N+c} \big|u(r)\big|^{q-2} u(r) u'(r) \, dr \\
&\leqslant \frac{p\omega_N}{N+\alpha-p} \int_0^\infty r^{N+c} \big|u(r)\big|^{q-1} \big|u'(r)\big| \, dr \\
&= \frac{p\omega_N}{N+\alpha-p} \int_0^\infty r^{N+c-\frac{N-1+\alpha}{p}} \big|u(r)\big|^{q-1} \big|u'(r)\big| r^{\frac{N-1+\alpha}{p}} \, dr \\
&\leqslant \frac{p\omega_N}{N+\alpha-p} \left(\int_0^\infty r^\alpha \big|u'(r)\big|^p r^{N-1} \, dr\right)^{\frac{1}{p}} \left(\int_0^\infty r^{(N+c-\frac{N-1+\alpha}{p})\frac{p}{p-1}} \big|u(r)\big|^{\frac{p(q-1)}{p-1}} \, dr\right)^{\frac{p-1}{p}} \\
&= \frac{p\omega_N^{\frac{p-1}{p}}}{N+\alpha-p} \left(\int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^p \, dx\right)^{\frac{1}{p}} \left(\int_0^\infty r^{N-1+c} \big|u(r)\big|^q r^{\frac{p-\alpha+c}{p-1}} \big|u(r)\big|^{\frac{q-p}{p-1}} \, dr\right)^{\frac{p-1}{p}} \\
&\leqslant \widehat{C}^{\frac{q-p}{p}} \frac{p\omega_N^{\frac{p-1}{p}}}{N+\alpha-p} \left(\int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^p \, dx\right)^{\frac{q}{p^2}} \left(\int_0^\infty r^{N-1+c} \big|u(r)\big|^q \, dr\right)^{\frac{p-1}{p}} \\
&= \widehat{C}^{\frac{q-p}{p}} \frac{p}{N+\alpha-p} \left(\int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^p \, dx\right)^{\frac{q}{p^2}} \left(\int_{\mathbb{R}^N} |x|^c |u|^q \, dx\right)^{\frac{p-1}{p}}
\end{aligned}$$

where the constant $\widehat{C}$ was given in Lemma 2.1. It follows that

$$\left(\int_{\mathbb{R}^N} |x|^c |u|^q \, dx\right)^{\frac{p}{q}} \leqslant \widetilde{C} \int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^p \, dx$$

where

$$\widetilde{C} = \omega_N^{-\frac{q-p}{q}} \left(\frac{p-1}{N+\alpha-p}\right)^{\frac{(p-1)(q-p)}{q}} \left(\frac{p}{N+\alpha-p}\right)^{\frac{p^2}{q}}$$

depends only on p, q, α and N. The proof is finished. □

We may regard (2.2) as an extension of the Caffarelli–Kohn–Nirenberg inequality [12], which was also considered in [16]. Our direct proof here follows the idea of [28] where the case that $\alpha = 0$ and $p = 2$ was considered. The above proof is also valid for the case $p = q$ and then a weighted version of the Hardy inequality [12] is obtained.

Lemma 2.3. *Let $1 < p < N + \alpha$. We have for all $u \in D_r^{1,p}(\mathbb{R}^N, |x|^\alpha)$,*

$$\int_{\mathbb{R}^N} |x|^{\alpha-p}|u|^p\,dx \leqslant \left(\frac{p}{N+\alpha-p}\right)^p \int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^p\,dx. \tag{2.3}$$

Now we give some radial inequalities which involve the conditions on A and V. First we note that under (A) and (V) the functional

$$\|u\|_{A,V} = \left(\int_{\mathbb{R}^N} A\big(|x|\big)|\nabla u|^p + V\big(|x|\big)|u|^p\,dx\right)^{\frac{1}{p}}$$

defines a norm so $W_r^{1,p}(\mathbb{R}^N; A, V)$ is well defined. In fact, we only need to show that $\|u\|_{A,V} = 0$ implies $u = 0$. Since $A(r) > 0$, if $\int_{\mathbb{R}^N} A(|x|)|\nabla u|^p\,dx = 0$, u is a constant. As $\liminf_{|x|\to\infty} V(|x|)|x|^{-a} > 0$, one has that $u = 0$.

Lemma 2.4. *Assume* (A) *and* (V) *with $a > -\frac{p(N-1)+\ell}{p-1}$. Then there exists $C > 0$ such that for all $u \in W_r^{1,p}(\mathbb{R}^N; A, V)$,*

$$\big|u(x)\big| \leqslant C|x|^{-\frac{p(N-1)+\ell+a(p-1)}{p^2}}\|u\|_{A,V}, \quad |x| \gg 1. \tag{2.4}$$

Proof. By a standard density argument, it suffices to prove (2.4) for $u \in C_{0,r}^\infty(\mathbb{R}^N)$. It follows from (A) and (V) that there exists $R > 0$ such that for some constant $C_0 > 0$,

$$A\big(|x|\big) \geqslant C_0|x|^\ell, \qquad V\big(|x|\big) \geqslant C_0|x|^a, \quad |x| \geqslant R.$$

For $u \in C_{0,r}^\infty(\mathbb{R}^N)$, when $\gamma > -(N-1)$, one has

$$\begin{aligned}\frac{d}{dr}\big(r^{\gamma+N-1}|u|^p\big) &= p\cdot r^{\gamma+N-1}|u|^{p-2}u\frac{du}{dr} + (\gamma+N-1)|u|^p r^{\gamma+N-2}\\ &\geqslant p\cdot r^{\gamma+N-1}|u|^{p-2}u\frac{du}{dr}.\end{aligned}$$

Take $\gamma = \frac{a(p-1)+\ell}{p}$, then for $r > R$,

$$\begin{aligned}
|u|^p r^{\gamma+N-1} &\leqslant p\int_r^\infty |u|^{p-1}s^{\gamma+N-1}\bigl|u'(s)\bigr|\,ds\\
&= p\int_r^\infty \bigl|u'(s)\bigr|s^{\frac{N-1+\ell}{p}}s^{\gamma-\ell+\frac{p-1}{p}(N-1+\ell)}|u|^{p-1}\,ds\\
&\leqslant p\left(\int_r^\infty \bigl|u'(s)\bigr|^p s^{N-1+\ell}\,ds\right)^{\frac{1}{p}}\left(\int_r^\infty s^{\frac{(\gamma-\ell)p}{p-1}+\ell}|u|^p s^{N-1}\,ds\right)^{\frac{p-1}{p}}\\
&\leqslant p\omega_N^{-1}\left(\int_{\mathbb{R}^N\setminus B_r(0)} |x|^\ell|\nabla u|^p\,dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^N\setminus B_r(0)} |x|^a|u|^p\,dx\right)^{\frac{p-1}{p}}\\
&\leqslant p\omega_N^{-1}C_0^{-1}\int_{\mathbb{R}^N} A\bigl(|x|\bigr)|\nabla u|^p+V\bigl(|x|\bigr)|u|^p\,dx.
\end{aligned}$$

It follows that

$$\bigl|u(x)\bigr|\leqslant C|x|^{-\frac{p(N-1)+\ell+a(p-1)}{p^2}}\|u\|_{A,V},\quad |x|>R. \qquad \square$$

Notice that $1<p<N+\ell$ implies $-\frac{p(N-1)+\ell}{p-1}<-p+\ell$ and

$$-\frac{p(N-1)+\ell+a(p-1)}{p^2}<-\frac{N+\ell-p}{p}\quad\Longleftrightarrow\quad a>-p+\ell,$$

therefore (2.4) is always valid by Lemma 2.1 with $\alpha=\ell$ for $-\frac{p(N-1)+\ell}{p-1}<a\leqslant -p+\ell$ and is more accurate for $a>-p+\ell$.

Lemma 2.5. *Assume* (A) *and* (V)*. Then there exists $r_0>0$ and $\widetilde{C}_0>0$ such that for all $u\in W_r^{1,p}(\mathbb{R}^N;A,V)\cap D_0^{1,p}(B_{r_0}(0),A)$,*

$$\bigl|u(x)\bigr|\leqslant \widetilde{C}_0|x|^{-\frac{p(N-1)+\ell_0+a_0(p-1)}{p^2}}\|u\|_{A,V},\quad 0<|x|\leqslant r_0, \tag{2.5}$$

where

$$D_0^{1,p}\bigl(B_{r_0}(0),A\bigr):=\left\{u\colon \int_{B_{r_0}(0)} A(x)|\nabla u|^p\,dx<\infty,\ u=0\ on\ \partial B_{r_0}(0)\right\}.$$

Proof. It follows from (A) and (V) that there exists $r_0>0$ such that for some constant $C_0>0$,

$$A\bigl(|x|\bigr)\geqslant C_0|x|^{\ell_0},\qquad V\bigl(|x|\bigr)\geqslant C_0|x|^{a_0},\quad 0<|x|\leqslant r_0.$$

For $u\in W_r^{1,p}(\mathbb{R}^N;A,V)\cap D_0^{1,p}(B_{r_0}(0),A)$, one has

$$\frac{d}{dr}\bigl(r^{\beta+N-1}|u|^p\bigr)=p\cdot r^{\beta+N-1}|u|^{p-2}u\frac{du}{dr}+(\beta+N-1)|u|^p r^{\beta+N-2}.$$

Thus for $0 < r \leqslant r_0$,

$$r^{\beta+N-1}|u|^p \leqslant p \int_r^{r_0} |u|^{p-1} s^{\beta+N-1} \left|u'(s)\right| ds - (\beta + N - 1) \int_r^{r_0} |u|^p s^{\beta+N-2}\, ds.$$

When $\beta \geqslant a_0 + 1$,

$$\begin{aligned}
\int_r^{r_0} |u|^p s^{\beta+N-2}\, ds &= \int_r^{r_0} |u|^p s^{a_0+N-1} s^{\beta-a_0-1}\, ds \\
&\leqslant \omega_N^{-1} r_0^{\beta-a_0-1} \int_{B_{r_0}(0)\setminus B_r(0)} |x|^{a_0}|u|^p\, dx \\
&\leqslant \omega_N^{-1} C_0^{-1} r_0^{\beta-a_0-1} \|u\|_{A,V}^p.
\end{aligned}$$

Now we take $\beta = \frac{a_0(p-1)+\ell_0}{p}$, then

$$\begin{aligned}
\int_r^{r_0} |u|^{p-1} s^{\beta+N-1}\left|u'(s)\right| ds &= \int_r^{r_0} \left|u'(s)\right| s^{\frac{N-1+\ell_0}{p}} s^{\beta-\ell_0+\frac{p-1}{p}(N-1+\ell_0)} |u|^{p-1}\, ds \\
&\leqslant \left(\int_r^{r_0} \left|u'(s)\right|^p s^{N-1+\ell_0}\, ds\right)^{\frac{1}{p}} \left(\int_r^{r_0} s^{\frac{p(\beta-\ell_0)}{p-1}+\ell_0} |u|^p s^{N-1}\, ds\right)^{\frac{p-1}{p}} \\
&\leqslant \omega_N^{-1} \left(\int_{B_{r_0}(0)\setminus B_r(0)} |x|^{\ell_0} |\nabla u|^p\, dx\right)^{\frac{1}{p}} \left(\int_{B_{r_0}(0)\setminus B_r(0)} |x|^{a_0}|u|^p\, dx\right)^{\frac{p-1}{p}} \\
&\leqslant \omega_N^{-1} C_0^{-1} \int_{\mathbb{R}^N} A\left(|x|\right)|\nabla u|^p + V\left(|x|\right)|u|^p\, dx \\
&= \omega_N^{-1} C_0^{-1} \|u\|_{A,V}^p.
\end{aligned}$$

Since

$$\beta + N - 1 \geqslant 0 \quad \Longleftrightarrow \quad a_0 \geqslant -\frac{p(N-1)+\ell_0}{p-1},$$

it follows that $\beta + N - 1 \leqslant 0$ implies $\beta - a_0 - 1 \geqslant \frac{N-p+\ell_0}{p-1} > 0$ provided $p < N + \ell_0$. Then from above arguments we have

$$\left|u(x)\right| \leqslant \widetilde{C}_0 |x|^{-\frac{p(N-1)+\ell_0+a_0(p-1)}{p^2}} \|u\|_{A,V}, \quad 0 < |x| \leqslant r_0,$$

where the constant $\widetilde{C}_0 = \widetilde{C}_0(a_0, r_0, N, p, \ell_0)$. $\square$

We make a comment on Lemmas 2.4 and 2.5 here. From the proofs of Lemmas 2.4 and 2.5 we see that for the special case $A(x) = |x|^\alpha$ and $V(x) = |x|^a$ with $a > -\frac{p(N-1)+\alpha}{p-1}$, the inequalities (2.4) and (2.5) can be unified as

$$|x|^{\frac{p(N-1)+\alpha+a(p-1)}{p^2}} \big|u(x)\big| \leqslant \big(p\omega_N^{-1}\big)^{\frac{1}{p}} \bigg(\int\limits_{\mathbb{R}^N} |x|^\alpha |\nabla u|^p + |x|^a |u|^p \, dx \bigg)^{\frac{1}{p}}$$

for $u \in W_r^{1,p}(\mathbb{R}^N, |x|^\alpha, |x|^a)$. This is an improvement of the classical Strauss radial lemma in [34] where this inequality was given on exterior spherical domain $|x| \geqslant 1$ for the case $p = 2$, $A = V = 1$, and of the work [29] where the case $p = 2$, $\alpha = 0$, $a \geqslant 0$ was considered.

3. Embedding results

In this section we establish the main embedding results in this paper. First we have

Lemma 3.1. *Assume* (A), (V) *and* (Q). *Let* $1 < p < N + \min\{\ell, \ell_0\}$, $1 \leqslant q \leqslant \infty$. *Then for any* $0 < r < R < \infty$, *the embedding*

$$W_r^{1,p}(B_R \setminus B_r; A, V) \hookrightarrow L^q(B_R \setminus B_r; Q)$$

is compact.

Proof. Given $0 < r < R < \infty$. Since $A(r) > 0$, $V(r) > 0$ are continuous on $(0, \infty)$, there are $C_R > 0$ and $C_r > 0$ such that

$$C_r \leqslant A\big(|x|\big) \leqslant C_R, \qquad C_r \leqslant V\big(|x|\big) \leqslant C_R, \quad x \in B_R \setminus B_r.$$

Therefore

$$\|u\|_{W_r^{1,p}(B_R \setminus B_r, A, V)} = \bigg(\int\limits_{B_R \setminus B_r} A\big(|x|\big)|\nabla u|^p + V\big(|x|\big)|u|^p \, dx \bigg)^{\frac{1}{p}}$$

is an equivalent norm of $W_r^{1,p}(B_R \setminus B_r) := \{u \in W^{1,p}(B_R \setminus B_r) \mid u \text{ is radial}\}$. By Ascoli Theorem, the embedding $W_r^{1,p}(B_R \setminus B_r) \hookrightarrow L^q(B_R \setminus B_r)$ is compact for all $1 \leqslant q \leqslant \infty$, and it is easy to see that the embedding $L^q(B_R \setminus B_r) \hookrightarrow L^q(B_R \setminus B_r; Q)$ is continuous as Q is bounded on $B_R \setminus B_r$, so that we have the conclusion. $\square$

Theorem 3.2. *Assume* (A), (V) *and* (Q). *Then we have the embedding*

$$W_r^{1,p}\big(\mathbb{R}^N; A, V\big) \hookrightarrow L^q\big(\mathbb{R}^N; Q\big)$$

for $q_* \leqslant q \leqslant q^*$ *when* $q^* < \infty$ *and for* $q_* \leqslant q < \infty$ *when* $q^* = \infty$. *Furthermore, the embedding is compact for* $q_* < q < q^*$.

Proof. To prove the embedding is continuous, it suffices to show

$$S_r(A, V, Q) := \inf_{u \in W_r^{1,p}(\mathbb{R}^N; A, V)} \frac{\int_{\mathbb{R}^N} A(|x|)|\nabla u|^p + V(|x|)|u|^p \, dx}{(\int_{\mathbb{R}^N} Q(|x|)|u|^q \, dx)^{\frac{p}{q}}} > 0. \tag{3.1}$$

Assume that there exists $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^N; A, V)$ such that

$$\int\limits_{\mathbb{R}^N} A(|x|)|\nabla u_n|^p + V(|x|)|u_n|^p\,dx = o(1) \quad \text{as } n \to \infty, \tag{3.2}$$

$$\int\limits_{\mathbb{R}^N} Q(|x|)|u_n|^q\,dx \equiv 1, \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

We will get a contradiction by showing

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} Q(|x|)|u_n|^q\,dx = 0. \tag{3.4}$$

Furthermore, to show the compactness of the embedding, we only need to show that for any $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^N; A, V)$,

$$u_n \rightharpoonup 0 \quad \text{in } W_r^{1,p}(\mathbb{R}^N; A, V) \tag{3.5}$$

implies (3.4).

By (A), (V) and (Q), there exist $R_0 > r_0 > 0$, for some $C_0 > 0$,

$$\begin{cases} A(|x|) \geqslant C_0|x|^{\ell}, & V(|x|) \geqslant C_0|x|^a, & Q(|x|) \leqslant C_0|x|^b, & \text{for } |x| \geqslant R_0, \\ A(|x|) \geqslant C_0|x|^{\ell_0}, & V(|x|) \geqslant C_0|x|^{a_0}, & Q(|x|) \leqslant C_0|x|^{b_0}, & \text{for } 0 < |x| \leqslant r_0. \end{cases}$$

For $R > R_0$ and $0 < r < r_0$, we estimate the integrals

$$\int\limits_{B_r} Q(|x|)|u|^q\,dx, \qquad \int\limits_{B_R^c} Q(|x|)|u|^q\,dx$$

for q be in different cases according to the definitions of q^* and q_*. In the following we will use C to denote various constants.

We first estimate $\int_{B_r} Q(|x|)|u|^q\,dx$. We will use a smooth cut-off function ϕ such that $\phi(x) = 1$ for $0 \leqslant |x| \leqslant \frac{r_0}{2}$, and $\phi(x) = 0$ for $|x| \geqslant r_0$.

Case 1.1. $a_0 \geqslant -p + \ell_0$, $b_0 \geqslant -p + \ell_0$. In this case

$$q^* = \frac{p(N + b_0)}{N + \ell_0 - p}.$$

Writing $q = \frac{p(N+c)}{N+\ell_0-p}$, by $q \leqslant q^*$, we see that $\alpha_1 := b_0 - c \geqslant 0$. Then with the function ϕ given above, by Lemma 2.2 with $\alpha = \ell_0$, for $r < \frac{r_0}{2}$,

$$\begin{aligned} \int\limits_{B_r} Q(|x|)|u|^q\,dx &\leqslant C_0 \int\limits_{B_r} |x|^{b_0}|u|^q\,dx \\ &\leqslant C_0 r^{b_0 - c} \int\limits_{B_r} |x|^c|u|^q\,dx \end{aligned}$$

$$
\begin{aligned}
&\leqslant C_0 r^{b_0-c} \int_{\mathbb{R}^N} |x|^c |\phi u|^q \, dx \\
&\leqslant C r^{b_0-c} \biggl(\int_{\mathbb{R}^N} |x|^{\ell_0} \bigl|\nabla(\phi u)\bigr|^p \, dx \biggr)^{\frac{q}{p}} \\
&\leqslant C r^{b_0-c} \biggl(\int_{\mathbb{R}^N} |x|^{\ell_0} \bigl(|\nabla u|^p |\phi|^p + |u|^p |\nabla \phi|^p\bigr) \, dx \biggr)^{\frac{q}{p}} \\
&= C r^{b_0-c} \biggl(\int_{B_{r_0}} |x|^{\ell_0} \bigl(|\nabla u|^p |\phi|^p + |\nabla \phi|^p |u|^p\bigr) \, dx \biggr)^{\frac{q}{p}} \\
&\leqslant C r^{b_0-c} \biggl(\int_{B_{r_0}} A\bigl(|x|\bigr) |\nabla u|^p + V\bigl(|x|\bigr) |u|^p \, dx \biggr)^{\frac{q}{p}} \\
&\leqslant C r^{b_0-c} \|u\|_{A,V}^q .
\end{aligned}
\tag{3.6}
$$

Here we have used a fact that for $\frac{r_0}{2} \leqslant |x| \leqslant r_0$, $|x|^{\ell_0} |\nabla \phi|^p \leqslant CV(|x|)$.

Case 1.2. $-p + \ell_0 \geqslant a_0 > -\frac{p(N-1)+\ell_0}{p-1}$, $b_0 \geqslant a_0$. In this case

$$
q^* = \frac{p^2(N-1+b_0) + p(\ell_0 - a_0)}{p(N-1+a_0) + (\ell_0 - a_0)}.
$$

Then $q \leqslant q^*$ implies

$$
\alpha_2 := b_0 - a_0 - (q-p) \frac{p(N-1) + \ell_0 + a_0(p-1)}{p^2} \geqslant 0.
$$

With the same function ϕ given above, we have by Lemma 2.5 that for $r < \frac{r_0}{2}$,

$$
\begin{aligned}
\int_{B_r} Q\bigl(|x|\bigr) |u|^q \, dx &\leqslant C_0 \int_{B_r} |x|^{b_0} |\phi u|^q \, dx \\
&= C_0 \int_{B_r} |x|^{b_0-a_0} |\phi u|^{q-p} |x|^{a_0} |\phi u|^p \, dx \\
&\leqslant C \|\phi u\|_{A,V}^{q-p} \int_{B_r} |x|^{b_0-a_0-(q-p)\frac{p(N-1)+\ell_0+a_0(p-1)}{p^2}} V\bigl(|x|\bigr) |u|^p \, dx \\
&\leqslant C r^{b_0-a_0-(q-p)\frac{p(N-1)+\ell_0+a_0(p-1)}{p^2}} \|u\|_{A,V}^q .
\end{aligned}
\tag{3.7}
$$

Case 1.3. $a_0 \leqslant -\frac{p(N-1)+\ell_0}{p-1}$, $b_0 \geqslant a_0$. In this case

$$
q^* = \infty.
$$

Then for any $q \geqslant p$, it always holds

$$\alpha_3 := b_0 - a_0 - (q-p)\frac{p(N-1)+\ell_0+a_0(p-1)}{p^2} \geqslant 0.$$

With the same function ϕ given above, we have by Lemma 2.5 that for $r < \frac{r_0}{2}$,

$$\begin{aligned}
\int_{B_r} Q\big(|x|\big)|u|^q\,dx &\leqslant C_0 \int_{B_r} |x|^{b_0}|\phi u|^q\,dx \\
&= C_0 \int_{B_r} |x|^{b_0-a_0}|\phi u|^{q-p}|x|^{a_0}|\phi u|^p\,dx \\
&\leqslant C\|\phi u\|_{A,V}^{q-p} \int_{B_r} |x|^{b_0-a_0-(q-p)\frac{p(N-1)+\ell_0+a_0(p-1)}{p^2}} V\big(|x|\big)|u|^p\,dx \\
&\leqslant C r^{b_0-a_0-(q-p)\frac{p(N-1)+\ell_0+a_0(p-1)}{p^2}} \|u\|_{A,V}^q.
\end{aligned} \tag{3.8}$$

Next we estimate $\int_{B_R^c} Q(|x|)|u|^q\,dx$.

Case 2.1. $b \geqslant a > -p+\ell$. In this case

$$q_* = \frac{p^2(N-1+b)+p\ell-ap}{p(N-1)+\ell+a(p-1)}.$$

Then $q \geqslant q_*$ implies that

$$\beta_1 := b - a - (q-p)\frac{p(N-1)+\ell+a(p-1)}{p^2} \leqslant 0.$$

Thus by Lemma 2.4, for $R > R_0$

$$\begin{aligned}
\int_{B_R^c} Q\big(|x|\big)|u|^q\,dx &\leqslant C_0 \int_{B_R^c} |x|^{b}|u|^q\,dx \\
&= C_0 \int_{B_R^c} |x|^{b-a}|u|^{q-p}|x|^{a}|u|^p\,dx \\
&\leqslant C\|u_n\|_{A,V}^{q-p} \int_{B_R^c} |x|^{b-a-(q-p)\frac{p(N-1)+\ell+a(p-1)}{p^2}} V\big(|x|\big)|u|^p\,dx \\
&\leqslant C R^{b-a-(q-p)\frac{p(N-1)+\ell+a(p-1)}{p^2}} \|u\|_{A,V}^q.
\end{aligned} \tag{3.9}$$

Case 2.2. $b \geqslant -p+\ell$, $a \leqslant -p+\ell$. In this case

$$q_* = \frac{p(N+b)}{N-p+\ell}.$$

Writing $q=\frac{p(N+c)}{N-p+\ell}$, then $q \geqslant q_*$ implies $\beta_2 := b-c \leqslant 0$. We choose a smooth cut-off function ψ such that $\psi(x)=0$ for $0 \leqslant |x| \leqslant R_0$, and $\phi(x)=1$ for $|x| \geqslant R_0+1$. Hence by Lemma 2.2 with $\alpha=\ell$, similar to the proof for (3.6), we have for $R > R_0+1$ that,

$$\begin{aligned}
\int_{B_R^c} Q\big(|x|\big)|u|^q\,dx &\leqslant C_0 \int_{B_R^c} |x|^{b-c}|x|^c|u|^q\,dx \\
&\leqslant C_0 R^{b-c} \int_{B_R^c} |x|^c|u|^q\,dx \\
&\leqslant C_0 R^{b-c} \int_{\mathbb{R}^N} |x|^c|\psi u|^q\,dx \\
&\leqslant C R^{b-c}\bigg(\int_{\mathbb{R}^N} |x|^\ell\big|\nabla(\psi u)\big|^p\,dx\bigg)^{\frac{q}{p}} \\
&\leqslant C R^{b-c}\bigg(\int_{\mathbb{R}^N} |x|^\ell\big(|\nabla u|^p|\psi|^p+|u|^p|\nabla\psi|^p\big)\,dx\bigg)^{\frac{q}{p}} \\
&= C R^{b-c}\bigg(\int_{B_{R_0}^c} |x|^\ell\big(|\nabla u|^p|\psi|^p+|\nabla\psi|^p|u|^p\big)\,dx\bigg)^{\frac{q}{p}} \\
&\leqslant C R^{b-c}\bigg(\int_{B_{R_0}^c} A\big(|x|\big)|\nabla u|^p+V\big(|x|\big)|u|^p\,dx\bigg)^{\frac{q}{p}} \\
&\leqslant C R^{b-c}\|u\|_{A,V}^q.
\end{aligned} \tag{3.10}$$

Case 2.3. $b \leqslant \max\{a, -p+\ell\}$ and $q \geqslant q_* := p$. When $a > -p+\ell$, $b \leqslant a$, it always holds

$$\beta_3 := b-a-(q-p)\frac{p(N-1)+\ell+a(p-1)}{p^2} \leqslant 0,$$

so we get similar to (3.9) that for $R > R_0$

$$\int_{B_R^c} Q\big(|x|\big)|u|^q\,dx \leqslant C R^{b-a-(q-p)\frac{p(N-1)+\ell+a(p-1)}{p^2}}\|u\|_{A,V}^q. \tag{3.11}$$

When $a \leqslant -p+\ell$, $b \leqslant -p+\ell$, then by writing $q=\frac{p(N+c)}{N+\ell-p}$, it always holds

$$\beta_4 := b-c \leqslant 0,$$

we have similar to (3.10) that

$$\int_{B_R^c} Q\big(|x|\big)|u|^q\,dx \leqslant C R^{b-c}\|u\|_{A,V}^q. \tag{3.12}$$

Now for $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^N; A, V)$ satisfying (3.2) and (3.3), we write

$$\int_{\mathbb{R}^N} Q\big(|x|\big)|u_n|^q\,dx = \int_{B_r} Q\big(|x|\big)|u_n|^q\,dx + \int_{B_R^c} Q\big(|x|\big)|u_n|^q\,dx + \int_{B_R \setminus B_r} Q\big(|x|\big)|u_n|^q\,dx$$

where $R > R_0$, $0 < r < r_0$. Since $q_* \leqslant q \leqslant q^* < \infty$ or $q_* \leqslant q < q^* = \infty$ imply

$$\alpha_i \geqslant 0 \quad (i = 1, 2, 3), \qquad \beta_j \leqslant 0 \quad (j = 1, 2, 3, 4),$$

it follows from (3.6)–(3.8), (3.9)–(3.12), Lemma 3.1 and (3.2) that (3.4) holds which contradicts (3.3). Therefore the embedding is continuous.

For any $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^N; A, V)$ satisfying (3.5), we can get (3.4) in a similar way by the fact that $q_* < q < q^*$ implies $\alpha_i > 0$ $(i = 1, 2, 3)$, $\beta_j < 0$ $(j = 1, 2, 3, 4)$. Therefore the embedding is compact. The proof is completed. □

Theorem 3.3. *Assume* (A), (V) *and* (Q) *with* $b_0 \leqslant a_0 < -\frac{p(N-1)+\ell_0}{p-1}$. *Then we have the embedding*

$$W_r^{1,p}\big(\mathbb{R}^N; A, V\big) \hookrightarrow L^q\big(\mathbb{R}^N; Q\big) \quad \text{for } \max\{q_*, q_{**}\} \leqslant q < \infty.$$

Furthermore, the embedding is compact for $\max\{q_*, q_{**}\} < q < \infty$.

Proof. We only need to estimate $\int_{B_r} Q(|x|)|u|^q\,dx$ for $0 < r \leqslant r_0$, where r_0 is given above, the other parts of the proof are similar to the corresponding ones of Theorem 3.2 and are omitted. It follows from $q \geqslant q_{**}$ that

$$\alpha_6 := b_0 - a_0 - (q-p)\frac{p(N-1) + \ell_0 + a_0(p-1)}{p^2} \geqslant 0.$$

With the same function ϕ given above, we have by Lemma 2.5, for $r < \frac{r_0}{2}$,

$$\int_{B_r} Q\big(|x|\big)|u|^q\,dx \leqslant C r^{b_0 - a_0 - (q-p)\frac{p(N-1)+\ell_0+a_0(p-1)}{p^2}} \|u\|_{A,V}^q.$$

The proof is completed. □

Theorem 3.4. *Assume* (A), (V) *and* (Q) *with* $b < \max\{a, -p + \ell\}$ *and* $b_0 > \min\{-p + \ell_0, a_0\}$ *Then the embedding*

$$W_r^{1,p}\big(\mathbb{R}^N; A, V\big) \hookrightarrow L^p\big(\mathbb{R}^N; Q\big)$$

is compact.

Proof. When $a_0 \geqslant -p + \ell_0$, $b_0 \geqslant \min\{-p + \ell_0, a_0\}$ implies

$$\alpha_4 := b_0 + p - \ell_0 \geqslant 0.$$

We take the same function ϕ as in Cases 1.$*$ of Theorem 3.2 and use the same argument for (3.6), then by Lemma 2.3, for $r < \frac{r_0}{2}$,

$$\begin{aligned}\int_{B_r} Q\big(|x|\big)|u|^p\,dx &\leqslant C_0 \int_{B_r} |x|^{b_0+p-\ell_0}|x|^{\ell_0-p}|\phi u|^p\,dx \\ &\leqslant C_0 r^{b_0+p-\ell_0} \int_{B_{r_0}} |x|^{\ell_0-p}|\phi u|^p\,dx \\ &\leqslant C r^{b_0+p-\ell_0} \int_{\mathbb{R}^N} |x|^{\ell_0}\big|\nabla(\phi u)\big|^p\,dx \\ &\leqslant C r^{b_0+p-\ell_0}\|u\|_{A,V}^p. \end{aligned} \tag{3.13}$$

When $a_0 \leqslant -p+\ell_0$,

$$\alpha_5 := b_0 - a_0 \geqslant 0.$$

Hence we have

$$\begin{aligned}\int_{B_r} Q\big(|x|\big)|u|^p\,dx &\leqslant C_0 \int_{B_r} |x|^{b_0-a_0}|x|^{a_0}|u|^p\,dx \\ &\leqslant C r^{b_0-a_0} \int_{B_r} V\big(|x|\big)|u|^p\,dx \\ &\leqslant C r^{b_0-a_0}\|u\|_{A,V}^p. \end{aligned} \tag{3.14}$$

When $a > -p+\ell$, $b \leqslant \max\{a, -p+\ell\}$ implies

$$\beta_5 := b - a \leqslant 0,$$

thus for $R > R_0$,

$$\begin{aligned}\int_{B_R^c} Q\big(|x|\big)|u|^p\,dx &\leqslant C_0 \int_{B_R^c} |x|^{b-a}|x|^{a}|u|^p\,dx \\ &\leqslant C R^{b-a} \int_{B_R^c} V\big(|x|\big)|u|^p\,dx \\ &\leqslant C R^{b-a}\|u\|_{A,V}^p. \end{aligned} \tag{3.15}$$

When $a \leqslant -p+\ell$, we have

$$\beta_6 := b + p - \ell \leqslant 0.$$

We take the same function ψ as above and use the same argument for (3.10). Then by Lemma 2.3,

$$\begin{aligned}\int\limits_{B_R^c} Q\big(|x|\big)|u|^p\,dx &\leqslant C_0 R^{b+p-\ell}\int\limits_{B_R^c}|x|^{\ell-p}|u|^p\,dx\\ &\leqslant CR^{b+p-\ell}\int\limits_{\mathbb{R}^N}|x|^\ell\big|\nabla(\psi u)\big|^p\,dx\\ &\leqslant CR^{b+p-\ell}\|u\|_{A,V}^p. \end{aligned}\tag{3.16}$$

The remaining part of the proof is similar to the one of Theorem 3.2 and we omit it. □

4. Existence results for Eq. (1.1)

In this section as an application of the embedding results in Section 3 we prove Theorem 1.1 by using Mountain Pass Theorem and its symmetric version [3]. In the following we denote

$$E := W_r^{1,p}\big(\mathbb{R}^N; A, V\big).$$

We first observe that a solution $u \in E$ of (1.1) corresponds a critical point of the functional

$$\Phi(u) = \frac{1}{p}\int\limits_{\mathbb{R}^N} A\big(|x|\big)|\nabla u|^p + V\big(|x|\big)|u|^p\,dx - \int\limits_{\mathbb{R}^N} Q\big(|x|\big)F(u)\,dx.$$

In fact, by (f_1), (f_3) and the embedding results built in Section 3, Φ is well defined on E and $\Phi \in C^1(E,\mathbb{R})$ with

$$\big\langle \Phi'(u), v\big\rangle = \int\limits_{\mathbb{R}^N} A\big(|x|\big)|\nabla u|^{p-2}\nabla u\nabla v + V\big(|x|\big)|u|^{p-2}uv\,dx - \int\limits_{\mathbb{R}^N} Q\big(|x|\big)f(u)v\,dx$$

for all $u, v \in E$.

We consider the case that q_* and q^* are well defined. The other cases are similar.

As $f(0) = 0$, (1.1) has a trivial solution $u \equiv 0$. By (f_3), we have for some $C > 0$,

$$F(u) \leqslant C\big(|u|^{q_1} + |u|^{q_2}\big), \quad \forall u \in \mathbb{R}.\tag{4.1}$$

Hence for $u \in E$, by Theorem 3.2, we have

$$\begin{aligned}\Phi(u) &= \frac{1}{p}\|u\|_E^p - \int\limits_{\mathbb{R}^N} Q\big(|x|\big)F(u)\,dx\\ &\geqslant \frac{1}{p}\|u\|_E^p - C\int\limits_{\mathbb{R}^N} Q\big(|x|\big)\big(|u|^{q_1}+|u|^{q_2}\big)\,dx\\ &\geqslant \frac{1}{p}\|u\|_E^p - C\big(\|u\|_E^{q_1}+\|u\|_E^{q_2}\big).\end{aligned}$$

Since $p \leqslant q_* < q_1 \leqslant q_2$, there exist $\rho > 0$ and $\eta > 0$ such that

$$\Phi(u) \geqslant \eta > 0 \quad \text{for all } u \in E \text{ with } \|u\|_E = \rho.\tag{4.2}$$

By (f_2), we deduce that

$$F(u) \geqslant C|u|^{\mu}, \quad \text{for } u \in \mathbb{R},\ |u| \geqslant 1. \tag{4.3}$$

From this and (4.1) we see that $q_* \leqslant \mu \leqslant q_2 < q^*$. For $u \in E$, by the embedding results again, it follows that $u \in L^{\mu}(\mathbb{R}^N, Q)$. Take a $u \in E$ such that $\int_{\{x:\ |u(x)| \geqslant 1\}} Q(|x|)|u|^{\mu}\,dx > 0$. Then

$$\begin{aligned}
\Phi(tu) &= \frac{1}{p}t^p\|u\|_E^p - \int_{\mathbb{R}^N} Q\big(|x|\big)F(tu)\,dx \\
&\leqslant \frac{1}{p}t^p\|u\|_E^p - Ct^{\mu}\int_{\{x:\ |u(x)| \geqslant 1\}} Q\big(|x|\big)|u|^{\mu}\,dx - \int_{\{x:\ |u(x)| \leqslant 1\}} Q\big(|x|\big)F(tu)\,dx \\
&\leqslant \frac{1}{p}t^p\|u\|_E^p - Ct^{\mu}\int_{\{x:\ |u(x)| \geqslant 1\}} Q\big(|x|\big)|u|^{\mu}\,dx \to -\infty \quad \text{as } t \to \infty.
\end{aligned}$$

Thus there is $e \in E$ with $\|e\|_E > \rho$ such that

$$\Phi(e) < 0. \tag{4.4}$$

Now we show that Φ satisfies the Palais–Smale condition. Let $\{u_n\} \subset E$ be such that

$$\Phi(u_n) \to c \in \mathbb{R}, \qquad \Phi'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{4.5}$$

Then for n large enough, we have by (f_2) that

$$\begin{aligned}
c + 1 + \|u_n\| &\geqslant \Phi(u_n) - \frac{1}{\mu}\big\langle \Phi'(u_n), u_n\big\rangle \\
&= \frac{\mu - p}{p\mu}\|u_n\|_E^p + \frac{1}{\mu}\int_{\mathbb{R}^N} Q\big(|x|\big)\big(u_n f(u_n) - \mu F(u_n)\big)\,dx \\
&\geqslant \frac{\mu - p}{p\mu}\|u_n\|_E^p.
\end{aligned}$$

Therefore $\{u_n\}$ is bounded in E. Up to a subsequence, we assume there exists $u \in E$ such that

$$u_n \rightharpoonup u \quad \text{in } E, \tag{4.6}$$

$$u_n(x) \to u(x) \quad \text{a.e. on } \mathbb{R}^N \tag{4.7}$$

and

$$u_n \to u \quad \text{in } L^q\big(\mathbb{R}^N, Q\big),\ \text{for all } q_* < q < q^* \tag{4.8}$$

by the compactness of the embedding. (4.8) and the assumptions on f imply that

$$\int_{\mathbb{R}^N} Q\big(|x|\big)\big(f(u_n) - f(u)\big)(u_n - u)\,dx \to 0, \quad n \to \infty. \tag{4.9}$$

Now we get from (4.5) and (4.6) that

$$\left\langle \Phi'(u_n) - \Phi'(u), u_n - u \right\rangle \to 0 \quad \text{as } n \to \infty$$

which and (4.9) imply

$$\int_{\mathbb{R}^N} A\big(|x|\big)\big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)(\nabla u_n - \nabla u)\,dx + \int_{\mathbb{R}^N} V\big(|x|\big)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(u_n - u)\,dx \to 0 \quad \text{as } n \to \infty. \tag{4.10}$$

We employ the following elementary inequalities: for all $\xi, \eta \in \mathbb{R}^N$, there is a constant $C = C(p) > 0$ such that

$$\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big)(\xi - \eta) \geqslant C|\xi - \eta|^p, \quad \text{for } p \geqslant 2,$$
$$\big(|\xi| + |\eta|\big)^{2-p}\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big)(\xi - \eta) \geqslant C|\xi - \eta|^2, \quad \text{for } 1 < p < 2.$$

When $p \geqslant 2$, we have for some $C > 0$

$$\int_{\mathbb{R}^N} A\big(|x|\big)\big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)(\nabla u_n - \nabla u)\,dx \geqslant C\int_{\mathbb{R}^N} A\big(|x|\big)\big|\nabla(u_n - u)\big|^p\,dx,$$
$$\int_{\mathbb{R}^N} V\big(|x|\big)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(u_n - u)\,dx \geqslant C\int_{\mathbb{R}^N} V\big(|x|\big)|u_n - u|^p\,dx.$$

When $1 < p < 2$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} A\big(|x|\big)\big|\nabla(u_n - u)\big|^p\,dx &\leqslant C\int_{\mathbb{R}^N} A\big(|x|\big)\big[\big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)\nabla(u_n - u)\big]^{\frac{p}{2}} \\
&\quad \times \big(|\nabla u_n| + |\nabla u|\big)^{\frac{p(2-p)}{2}}\,dx \\
&\leqslant C\left(\int_{\mathbb{R}^N} A\big(|x|\big)\big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)\nabla(u_n - u)\,dx\right)^{\frac{p}{2}} \\
&\quad \times \left(\int_{\mathbb{R}^N} A\big(|x|\big)\big(|\nabla u_n|^p + |\nabla u|^p\big)\,dx\right)^{\frac{2-p}{p}} \\
&\leqslant C\left(\int_{\mathbb{R}^N} A\big(|x|\big)\big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)\nabla(u_n - u)\,dx\right)^{\frac{p}{2}},
\end{aligned}$$

$$\int_{\mathbb{R}^N} V\big(|x|\big)|u_n - u|^p\,dx \leqslant C\int_{\mathbb{R}^N} A\big(|x|\big)\big[\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(u_n - u)\big]^{\frac{p}{2}}\big(|u_n| + |u|\big)^{\frac{p(2-p)}{2}}\,dx$$

$$\leqslant C\bigg(\int_{\mathbb{R}^N} V\big(|x|\big)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(u_n - u)\,dx\bigg)^{\frac{p}{2}}$$

$$\times\bigg(\int_{\mathbb{R}^N} V\big(|x|\big)\big(|u_n|^p + |u|^p\big)\,dx\bigg)^{\frac{2-p}{p}}$$

$$\leqslant C\bigg(\int_{\mathbb{R}^N} V\big(|x|\big)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(u_n - u)\,dx\bigg)^{\frac{p}{2}}.$$

It follows from (4.10) that

$$\|u_n - u\|_E \to 0 \quad \text{as } n \to \infty$$

and so Φ satisfies the Palais–Smale condition.

By Mountain Pass Theorem [3], Φ has a critical point $u^* \in W_r^{1,p}(\mathbb{R}^N; A, V)$ with $\Phi(u^*) \geqslant \eta > 0$ which gives a nontrivial radial solution of (1.1).

When (f_4) is verified, the functional Φ is even on E. By (f_3) and (f_5) we have that $q_* < \tau < q^*$. Hence by the embedding $E \hookrightarrow L^\tau(\mathbb{R}^N; Q)$, we see that $u \in E$ implies $u \in L^\tau(\mathbb{R}^N; Q)$. Now for $u \in E$, by (f_5) we get

$$\Phi(u) = \frac{1}{p}\|u\|_E^p - \int_{\mathbb{R}^N} Q\big(|x|\big)F(u)\,dx$$

$$\leqslant \frac{1}{p}\|u\|_E^p - C\|u\|_{L^\tau(\mathbb{R}^N; Q)}^\tau. \tag{4.11}$$

For any a finite-dimensional subspace $F \subset E$, since all norms on F are equivalent, it follows from (4.11) and $\tau > p$ that there is $R_F > 0$ such that

$$\max_{u \in F,\ \|u\|_E = R_F} \Phi(u) < 0.$$

By the Symmetric Mountain Pass Theorem [3], we get that (1.1) has infinitely many solutions in $W_r^{1,p}(\mathbb{R}^N; A, V)$. This completes the proof of Theorem 1.1.

Acknowledgment

The authors would like to thank the referee for carefully reading the manuscript, giving valuable comments and suggestions to improve the exposition of the paper.

References

[1] A. Ambrosetti, V. Felli, A. Malchiodi, Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity, J. Eur. Math. Soc. (JEMS) 7 (2005) 117–144.
[2] A. Ambrosetti, A. Malchiodi, D. Ruiz, Bounded states of nonlinear Schrödinger equations with potentials vanishing at infinity, J. Anal. Math. 98 (2006) 317–348.
[3] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.

[4] A. Ambrosetti, Z.-Q. Wang, Nonlinear Schrödinger equations with vanishing and decaying potentials, Differential Integral Equations 18 (2005) 1321–1332.
[5] M. Badiale, S. Rolando, A note on nonlinear elliptic problems with singular potentials, Rend. Lincei Mat. Appl. 17 (2006) 1–13.
[6] T. Bartsch, Z.-Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbf{R}^N$, Comm. Partial Differential Equations 20 (1995) 1725–1741.
[7] H. Berestycki, P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, Arch. Ration. Mech. Anal. 82 (1983) 313–345.
[8] J. Byeon, Z.-Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations, Arch. Ration. Mech. Anal. 165 (2002) 295–316.
[9] J. Byeon, Z.-Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations, II, Calc. Var. Partial Differential Equations 18 (2003) 207–219.
[10] J. Byeon, Z.-Q. Wang, Spherical semiclassical states of a critical frequency for Schrödinger equations with decaying potential, J. Eur. Math. Soc. (JEMS) 8 (2006) 217–228.
[11] J. Byeon, Z.-Q. Wang, Standing waves for nonlinear Schrödinger equations with singular potentials, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009) 943–958.
[12] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequalities with weights, Compos. Math. 53 (1984) 259–275.
[13] M. Conti, S. Crotti, D. Pardo, On the existence of positive solutions for a class of singular elliptic equations, Adv. Differential Equations 3 (1998) 111–132.
[14] M. Conti, S. Terracini, G. Verzini, Nodal solutions to a class of nonstandard superlinear equations on $\mathbb{R}^N$, Adv. Differential Equations 7 (2002) 297–318.
[15] F. Catrina, Z.-Q. Wang, On the Caffarelli–Kohn–Nirenberg inequalities: sharp constants, existence (and non-existence) and symmetry extremal functions, Comm. Pure Appl. Math. 54 (2001) 229–258.
[16] F. Catrina, Z.-Q. Wang, An one-dimensional nonlinear degenerate elliptic equation, in: Electron. J. Differ. Equ. Conf., vol. 06, 2000, pp. 89–99.
[17] D.G. Costa, On a class of elliptic systems in $\mathbf{R}^N$, Electron. J. Differential Equations 07 (1994) 1–14.
[18] W. Ding, W.-M. Ni, On the existence of positive entire solutions of a semilinear elliptic equation, Arch. Ration. Mech. Anal. 91 (1986) 283–308.
[19] N. Ghoussoub, C. Yuan, Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents, Trans. Amer. Math. Soc. 352 (2000) 5703–5743.
[20] V. Glaser, A. Martin, H. Grosse, W. Thirring, A family of optimal conditions for the absence of bound states in a potential, in: Stud. Math. Phys., Princeton University Press, 1976, pp. 169–194.
[21] E. Lieb, Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities, Ann. of Math. 118 (1983) 349–374.
[22] V.G. Maz'ja, Sobolev Spaces, Springer-Verlag, Berlin, 1985.
[23] W.-M. Ni, A nonlinear Dirichlet problem on the unit ball and its applications, Indiana Univ. Math. J. 31 (1982) 801–807.
[24] E.S. Noussair, C.A. Swanson, Decaying solutions of semilinear elliptic equations in $\mathbf{R}^N$, SIAM J. Math. Anal. 20 (1989) 1336–1343.
[25] W. Omana, M. Willem, Homoclinic orbits for a class of Hamiltonian systems, Differential Integral Equations 5 (1992) 1115–1120.
[26] B. Opic, A. Kufner, Hardy-Type Inequalities, Pitman Res. Notes Math. Ser., vol. 219, Longman Scientific and Technical, Harlow, 1990.
[27] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. 43 (1992) 270–291.
[28] W. Rother, Some existence results for the equation $\Delta U + K(x)U^p = 0$, Comm. Partial Differential Equations 15 (1990) 1461–1473.
[29] P. Sintzoff, M. Willem, A semilinear elliptic equation on $\mathbf{R}^N$ with unbounded coefficients, in: Variational and topological methods in the study of nonlinear phenomena, Pisa, 2000, in: Progr. Nonlinear Differential Equations Appl., vol. 49, Birkhäuser, Boston, 2002, pp. 105–113.
[30] B. Sirakov, Existence and multiplicity of solutions of semi-linear elliptic equations in $\mathbb{R}^N$, Calc. Var. Partial Differential Equations 11 (2000) 119–142.
[31] J. Su, Z.-Q. Wang, M. Willem, Nonlinear Schrödinger equations with unbounded and decaying radial potentials, Commun. Contemp. Math. 9 (2007) 571–583.
[32] J. Su, Z.-Q. Wang, M. Willem, Weighted Sobolev embedding with unbounded and decaying radial potentials, J. Differential Equations 238 (2007) 201–219.
[33] P. Souplet, Q. Zhang, Stability for semilinear parabolic equations with decaying potentials in $\mathbf{R}^n$ and dynamical approach to the existence of ground states, Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002) 683–703.
[34] W.A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55 (1977) 149–162.
[35] S. Terracini, On positive entire solutions to a class of equations with a singular coefficient and critical exponent, Adv. Differential Equations 1 (1996) 241–264.